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CONSISTENCY OF RANDOM FIELD SPECIFICATIONS

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Abstract

A random field specification is a consistent family of conditional probability distributions parametrized by a directed set $\mathcal{A}$. For a subset $\mathcal{B} \subset \mathcal{A}$ there is the problem of determining which, if any, specifications arise from a given family of conditional probability candidates parametrized by $\mathcal{B}$. For an algebraic form of this problem we give necessary and sufficient conditions for existence and uniqueness. We apply the results to one-dimensional random fields with nearest neighbour constraints.

1. Introduction

Certain problems in the theory of random fields had their origins in lattice gas models of statistical mechanics. The basic structures are a countably infinite set of sites S and a finite set Y which describes the configuration of a single site, the overall configuration being described by a point of $X = Y^S$. In applications to magnetic phenomena, S corresponds to the locations of atoms in a crystal and the point of Y gives the direction of the spin of a specific atom. In the physical model one prescribes an interaction potential among sites and determines the properties of probability measures on X which are consistent with the given potential (for details see Ruelle [7]).

Dobrushin [3] considered systems of this type in terms of conditional probabilities. For a given probability measure on X and a given $\Lambda \subset S$ one can consider the conditional probability distribution of the configuration of the Λ sites given the configuration of the remaining sites. The finite set conditional probabilities are those for all finite $\Lambda \subset S$; the one point conditional probabilities are those for $\Lambda = \{j\}, j \in S$. Dobrushin considered the problem of determining the probability measures on X corresponding to a specified family of finite set conditional probabilities. This line of investigation has been extensively pursued as the theory of specifications of random fields (see Preston [6]).

It is natural to consider the relationship between the conditional probability approach and earlier studies based on potentials. Under assumption of positivity, together with regularity conditions, it has been shown that the one point conditional probabilities determine the finite set conditional probabilities and a corresponding potential can be constructed (see [2], [5], [9]).

In both the physical and mathematical contexts it is natural to consider models in which positivity fails. To deal with this case we consider a set of *allowed* configurations $X_0 \subset X$ where positivity does obtain. For a given X_0 and a specified collection of conditional probability candidates we ask two questions. Are the given candidates sufficient to determine the finite set conditional probabilities? Can this be done in a consistent manner?

In order to avoid certain technical difficulties we recast these problems in algebraic form. In this formulation the answers to the two questions posed above can be expressed in terms of criteria involving concepts from homological algebra. This still leaves the problem of expressing the geometric constraints of a given problem in terms of the algebraic criteria. For an important class of one dimensional systems we can give a reasonably satisfactory solution. Higher dimensional models pose difficult combinatorial problems.

2. Algebraic and Ratio Specifications

The basic structures for our algebraic approach are as follows. We have a set X_o and a collection of partitions of X_o into equivalence classes. The partitions are parametrized by the set $\mathcal{A}$, which itself is a collection of subsets of the set S. For each $\alpha \in \mathcal{A}$ and $x \in X_o$ we write $\{x\}_\alpha$ to denote the equivalence class of x corresponding to α; also we write $x = y \bmod \alpha$ for this equivalence. We require that $\{x\}_\alpha \subset \{x\}_\beta$ when $\alpha \subset \beta$ with $x \in X_o$, $\alpha, \beta \in \mathcal{A}$. Also we require that for each $\alpha, \beta \in \mathcal{A}$ there is a $\gamma \in \mathcal{A}$ with $\alpha \cup \beta \subset \gamma$. The above sets and equivalence classes will be denoted by the symbol $\mathcal{X}$.

2.1. *Definition.* An *algebraic specification* $P_\alpha(x,y)$ is a real valued function defined for all $\alpha \in \mathcal{A}$ and $x = y \bmod \alpha$ which satisfies

$$P_\alpha(x,y) > 0, \tag{1}$$

$$P_\alpha(x,y) = P_\alpha(x,x), \tag{2}$$

$$P_\alpha(x,y)\, P_\beta(y,x) = P_\beta(x,y)\, P_\alpha(y,x) \text{ when } \alpha \subset \beta \tag{3}$$

for all $\alpha, \beta \in \mathcal{A}$ and $x = y \bmod \alpha$.

This definition is motivated by conditional probabilities: (2) corresponds to measurability with respect to the appropriate σ-field and (3) corresponds to the consistency requirement (see Preston [6], Lemma 5.1).

2.2. *Definition.* Two algebraic specifications $P_\alpha(x,y)$ and $Q_\alpha(x,y)$ on $\mathcal{X}$ are said to be *equivalent* if

$$P_\alpha(x,y)\, Q_\alpha(y,x) = P_\alpha(y,x)\, Q_\alpha(x,y)$$

for all $\alpha \in \mathcal{A}$ and $x = y \bmod \alpha$.

In the conditional probability context one can often choose a particular element of each equivalence class by the requirement of normalization with respect to certain measures.

2.3. *Definition.* A *ratio specification* $F_\alpha(x,y)$ on $\mathcal{X}$ is a real valued function defined for all $\alpha \in \mathcal{A}$ and $x = y \bmod \alpha$ which satisfies

$$F_\alpha(x,y) > 0, \tag{4}$$

$$F_\alpha(x,y)\, F_\alpha(y,z) = F_\alpha(x,z), \tag{5}$$

$$F_\alpha(x,y) = F_\beta(x,y) \quad \text{when } \alpha \subset \beta \tag{6}$$

for all $\alpha,\ \beta \in \mathcal{A}$ and $x = y \bmod \alpha$, $y = z \bmod \alpha$.

2.4. LEMMA. *An algebraic specification* $P_\alpha(x,y)$ *on* $\mathcal{X}$ *determines uniquely a ratio specification. A ratio specification* $F_\alpha(x,y)$ *on* $\mathcal{X}$ *determines an algebraic specification up to equivalence.*

Proof. Let $P_\alpha(x,y)$ be an algebraic specification on $\mathcal{X}$. Define $F_\alpha(x,y) = P_\alpha(x,y)/P_\alpha(y,x)$. Then (4) follows from (1) while (6) follows from (1) and (3). Also (5) follows from (1) and (2). To prove the second part we select an element $z(\alpha,x)$ from each equivalence class $\{x\}_\alpha$, *i.e.* $z(\alpha,x) = x \bmod \alpha$ and $x = y \bmod \alpha$ implies $z(\alpha,x) = z(\alpha,y)$. Now given the ratio specification $F_\alpha(x,y)$, define $P_\alpha(x,y) = F_\alpha(x,z(\alpha,x))$ for $x = y \bmod \alpha$. Then (1) and (2) follow from this definition and (4). We have $P_\alpha(x,y)/P_\alpha(y,x) = F_\alpha(x,y)$ by (5), so (6) then implies (3). It is not difficult to verify that different choices of $z(\alpha,x)$ give equivalent algebraic specifications and that equivalent algebraic specifications yield the same ratio specification.

Now we come to the basic problems. Assume that we are given $\mathcal{B} \subset \mathcal{A}$ and a function $P_\alpha(x,y)$ satisfying (1) and (2) for all $\alpha \in \mathcal{B}$ and $x = y \bmod \alpha$. Does there exist an extension of $P_\alpha(x,y)$ which is an algebraic specification on $\mathcal{X}$? Can there be nonequivalent extensions? We shall pose and answer these questions for ratio specifications because certain concepts from homological algebra arise naturally in this context.

For any $\mathcal{B} \subset \mathcal{A}$ let $C(\mathcal{B})$ denote the free abelian group generated by triples of the form $(y,\ z,\ \alpha)$ where $\alpha \in \mathcal{B}$ and $y = z \bmod \alpha$. Let $C(X_0)$ denote the free abelian group generated by the elements of X_0. We define the boundary homomorphism

$$\partial_{\mathcal{B}} \ :\ C(\mathcal{B}) \to C(X_0) \qquad \text{by}$$

$$\partial_{\mathcal{B}}\Big(\sum_{i=1}^{n} k_i\,(x_i, y_i, \alpha_i)\Big) = \sum_{i=1}^{n} (k_i x_i - k_i y_i) \tag{7}$$

where the k_i's are integers. Elements in $\ker \partial_{\mathcal{B}}$ will be called *cycles*. Given a function $F_\alpha(x,y)$ defined for $\alpha \in \mathcal{B}$ and $x = y \bmod \alpha$ which satisfies (4), we can extend it to a homomorphism $F_{\mathcal{B}} : C(\mathcal{B}) \to R^+$, the positive real numbers under multiplication, by

$$F_{\mathcal{B}}\Big(\sum_{i=1}^{n} k_i(x_i, y_i, \alpha_i)\Big) = \prod_{i=1}^{n} \big(F_{\alpha_i}(x_i, y_i)\big)^{k_i}\ . \tag{8}$$

2.5. LEMMA. $F_{\mathcal{A}}$ *is trivial on* $\ker \partial_{\mathcal{A}}$ *if and only if* $F_\alpha(x,y)$ *satisfies* (5) *and* (6).

Proof. Suppose $F_{\mathcal{A}}$ is trivial on cycles. When $x = y \bmod \alpha$, $y = z \bmod \alpha$ and $\alpha \subset \beta$, we have the cycles $(x,y,\alpha) + (y,z,\alpha) - (x,z,\alpha)$ and $(x,y,\alpha) - (x,y,\beta)$. Apply $F_{\mathcal{A}}$ to these cycles to obtain (5) and (6). Conversely suppose (5) and (6) hold for $F_\alpha(x,y)$. Cycles of the form $\sum_{i=1}^{n} (x_i,\ x_{i+1},\ \alpha_i)$ with $x_1 = x_{n+1}$ generate $\ker \partial_{\mathcal{A}}$. By our basic assumptions on $\mathcal{X}$ there is a $\gamma \in \mathcal{A}$ containing $\alpha_1, \ldots, \alpha_n$. Using this γ and (5), it follows easily from (6) that $F_{\mathcal{A}}$ is trivial on cycles of this form. Since $F_{\mathcal{A}}$ is a homomorphism and these cycles generate $\ker \partial_{\mathcal{A}}$, we have the desired result.

2.6. THEOREM. *Let* $F_\alpha(x,y)$ *be a real valued function defined for* $\alpha \in \mathcal{B}$ *and* $x = y \bmod \alpha$ *which satisfies* (4). *Then* $F_\alpha(x,y)$ *is the restriction to* $\mathcal{B}$ *of a ratio specification on* $\mathcal{X}$ *if and only if* $F_{\mathcal{B}}$ *is trivial on* $\ker \partial_{\mathcal{B}}$.

Proof. Let $i : C(\mathcal{B}) \to C(\mathcal{A})$ be the natural inclusion homomorphism. Then

$$\partial_{\mathcal{B}} = \partial_{\mathcal{A}} \circ i.$$

Thus there exists a monomorphism

$$k : C(\mathcal{B})/\ker \partial_{\mathcal{B}} \to C(\mathcal{A})/\ker \partial_{\mathcal{A}}$$

given by $k(a + \ker \partial_{\mathcal{B}}) = i(a) + \ker \partial_{\mathcal{A}}$. Suppose that $F_{\mathcal{B}}$ is trivial on $\ker \partial_{\mathcal{B}}$. Then we have the induced homomorphism

$$\tilde{F}_{\mathcal{B}} \ : \ C(\mathcal{B})/\ker \partial_{\mathcal{B}} \to R^+ .$$

Since the positive reals under multiplication is a divisible abelian group and k is a monomorphism, there exists a homomorphism

$$\tilde{F}_{\mathcal{A}} \ : \ C(\mathcal{A})/\ker \partial_{\mathcal{A}} \to R^+ .$$

such that $\tilde{F}_{\mathcal{B}} = \tilde{F}_{\mathcal{A}} \circ k$ (see Theorem 21.1 of Fuchs [4]). By combining $\tilde{F}_{\mathcal{A}}$ with the quotient homomorphism $C(\mathcal{A}) \to C(\mathcal{A})/\ker \partial_{\mathcal{A}}$ we obtain a homomorphism $F_{\mathcal{A}} : C(\mathcal{A}) \to R^+$. By construction $F_{\mathcal{A}}$ is trivial on $\ker \partial_{\mathcal{A}}$ and $F_{\mathcal{B}} = F_{\mathcal{A}} \circ i$. Lemma 2.5 shows that restricting $F_{\mathcal{A}}$ to generators provides the required ratio specification on $\mathcal{X}$.

Conversely, suppose $F_\alpha(x,y)$ is the restriction to $\mathcal{B}$ of a ratio specification on $\mathcal{X}$. We have the following commutative diagram:

$$\begin{tikzcd}
& R^+ & \\
C(\mathcal{B}) \arrow[ur, "F_{\mathcal{B}}"] \arrow[rr, "i"] \arrow[dr, "\partial_{\mathcal{B}}"'] & & C(\mathcal{A}) \arrow[ul, "F_{\mathcal{A}}"'] \arrow[dl, "\partial_{\mathcal{A}}"] \\
& C(X_o) &
\end{tikzcd}$$

Hence to show that $F_{\mathcal{B}}$ is trivial on $\ker \partial_{\mathcal{B}}$ it is sufficient to show that $F_{\mathcal{A}}$ is trivial on $\ker \partial_{\mathcal{A}}$. However, $F_{\mathcal{A}}$ comes from a ratio specification on $\mathcal{X}$, so by Lemma 2.5 it is trivial on $\ker \partial_{\mathcal{A}}$.

The next theorem describes all extensions of $F_{\mathcal{B}}$ to ratio specification on $\mathcal{X}$. First we need

2.7. *Definition.* For $\mathcal{B} \subset \mathcal{A}$ we say that x *is connected to* y mod $\mathcal{B}$ if there exist $x_o, x_1, \dots, x_n \in X_o$, $\beta_1, \dots, \beta_n \in \mathcal{B}$ with $x_o = x$, $x_n = y$ and $x_i = x_{i-1}$ mod β_i, $1 \le i \le n$.

Being connected mod $\mathcal{B}$ is an equivalence relation on X_o; we shall write $\{x\}_{\mathcal{B}}$ for the class of elements connected to x mod $\mathcal{B}$. Note that the requirements on $\mathcal{A}$ imply that $x = y$ mod $\mathcal{A}$ if and only if there exists $\alpha \in \mathcal{A}$ with

$x = y$ mod α.

Choose a single representative from each $\mathcal{B}$ equivalence class and let the set of these chosen elements be denoted $X_{\mathcal{B}}$. We can choose representatives of the $\mathcal{A}$ equivalence classes so that $X_{\mathcal{A}} \subset X_{\mathcal{B}}$.

2.8. *Definition.* We denote by $\mathcal{G}$ the set of all positive real valued functions on $X_{\mathcal{B}}$. Two functions $f, g \in \mathcal{G}$ are called *equivalent* if $f(x)/f(y) = g(x)/g(y)$ for all $x, y \in X_{\mathcal{B}}$ with $x = y$ mod $\mathcal{A}$. We use $\mathcal{F}$ to denote the equivalence classes of $\mathcal{G}$ under this relation.

2.9. THEOREM. *Let* $F_\alpha(x,y)$ *be a positive real valued function defined for* $\alpha \in \mathcal{B}$ *and* $x = y$ mod α. *Suppose* $F_{\mathcal{B}}$ *defined by* (8) *is trivial on* $\ker \partial_{\mathcal{B}}$. *Then there is a one-to-one correspondence between* $\mathcal{F}$ *and extensions of* $F_{\mathcal{B}}$ *which come from ratio specifications on* $\mathcal{X}$.

Proof. Any extension of $F_{\mathcal{B}}$ to a ratio specification will satisfy certain conditions. In particular if $x = y$ mod $\mathcal{B}$, then there is some $\alpha \in \mathcal{A}$ with $x = y$ mod α and $F_\alpha(x,y)$ is uniquely determined. Thus we can assume without loss of generality that we are given $F_\alpha(x,y)$ satisfying (6) wherever $x = y$ mod $\mathcal{B}$ and $x = y$ mod α. Now given $f \in \mathcal{G}$ and $(w,z,\alpha) \in C(\mathcal{A})$ we define

$$F_\alpha(w,z) = F_\gamma(w,x)\, F_\gamma(y,z)\, f(x)/f(y) \tag{9}$$

where $x,y \in X_{\mathcal{B}}$, $w = x$ mod $\mathcal{B}$, $y = z$ mod $\mathcal{B}$, $\alpha \subset \gamma$, $w = x$ mod γ, $y = z$ mod γ. Note that (9) is independent of the choice of γ satisfying the above. It is straightforward to verify that $F_\alpha(w,z)$ so defined satisfies (4), (5) and (6) and thus gives a ratio specification on $\mathcal{X}$ which extends that given. From the defining formula (9) it follows that equivalent elements of $\mathcal{G}$ yield the same ratio specification, while nonequivalent elements of $\mathcal{G}$ yield distinct ratio specifications.

Finally given a ratio specification $F_\alpha(x,y)$ we define $f \in \mathcal{G}$ by

$$f(x) = F_\alpha(x, y) \tag{10}$$

where $x \in X_{\mathfrak{B}}$, $y \in X_{\mathcal{A}}$ and $x = y \bmod \alpha$. Since the y in $X_{\mathcal{A}}$ equivalent to x is unique and (6) holds, (10) is well defined. A calculation shows that the f so defined satisfies (9).

Note that $\mathcal{F}$ consists of a single element exactly when $\{x\}_{\mathcal{A}} = \{x\}_{\mathfrak{B}}$ for each $x \in X_o$.

2.10. COROLLARY. *Assume* $F_{\mathfrak{B}}$ *is trivial on* $\ker \partial_{\mathfrak{B}}$. *Then the extension of* $F_{\mathfrak{B}}$ *to a ratio specification on* $\mathcal{X}$ *is unique if and only if* $\{x\}_{\mathcal{A}} = \{x\}_{\mathfrak{B}}$ *for all* $x \in X_o$, i.e. *if and only if whenever* x, y *are connected* mod $\mathcal{A}$, *they are connected* mod $\mathfrak{B}$.

2.11. *Definition.* The *length* of the cycle $\sum_{i=1}^{n} k_i(z_i, y_i, \alpha_i) \in \ker \partial_{\mathfrak{B}}$ is $\sum_{i=1}^{n} |k_i|$.

2.12. *Remark.* Let $\tilde{C}(\mathfrak{B})$ be the free abelian group generated by pairs (x,y) where $x = y \bmod \alpha$ for $\alpha \in \mathfrak{B}$. We have the mapping $\phi : C(\mathfrak{B}) \to \tilde{C}(\mathfrak{B})$ with

$$\phi\left(\sum_{i=1}^{n} k_i(x_i, y_i, \alpha_i)\right) = \sum_{i=1}^{n} k_i(x_i, y_i).$$

Thus we have the boundary operator $\tilde{\partial}_{\mathfrak{B}} : \tilde{C}(\mathfrak{B}) \to C(X)$ satisfying $\partial_{\mathfrak{B}} = \tilde{\partial}_{\mathfrak{B}} \circ \phi$. The homomorphism ϕ is onto and thus induces an isomorphism from $\ker \partial_{\mathfrak{B}} / \ker \phi$ to $\ker \tilde{\partial}_{\mathfrak{B}}$. A set of generators for $\ker \tilde{\partial}_{\mathfrak{B}}$ thus provides a set of generators for $\ker \partial_{\mathfrak{B}} / \ker \phi$. We can pick representatives for these in $\ker \partial_{\mathfrak{B}}$ in such a way that the cycle length is preserved. These representatives, together with a generating set for $\ker \phi$ will generate $\ker \partial_{\mathfrak{B}}$. Finally we note that $\ker \phi$ is generated by cycles of the form $(x,y,\alpha_1) - (x,y,\alpha_2)$.

3. APPLICATION TO RANDOM FIELDS

We now relate the algebraic formalism of the previous section to the model originally introduced. Recall that S is a countably infinite set, Y a finite set and $X = Y^S$. There is very little additional effort required to allow a different Y at each site, but for simplicity of notation we shall not do this. The set of allowed configurations for which we want our conditional probabilities positive is denoted X_o. In most cases considered in the literature X_o is obtained from X by exclusion rules which involve sites at finite distances from each other. The set $\mathcal{A}$ is the set of all finite subsets of S. For $\alpha \in \mathcal{A}$ and $x,y \in X_o$, $x = y \bmod \alpha$ if $x^j = y^j$ for all $j \in S \setminus \alpha$. We use superscripts to denote components.

For a given X_o and $\mathfrak{B} \subset \mathcal{A}$ we wish to know whether conditional probabilities given for $\alpha \in \mathfrak{B}$ determine the finite set conditional probabilities (*i.e.* those for $\mathcal{A}$) and a set of generators for $\ker \partial_{\mathfrak{B}}$ so we may express consistency conditions. For the case $X_o = X$, if $\mathfrak{B}$ contains all singletons $\{j\}$, $j \in S$, corresponding to one point conditional probabilities, then the finite set conditional probabilities can be computed (see [8]). Also cycles of length 4 are sufficient (see [8]). One needs, in addition, some regularity conditions; we shall express one form of these in a result below.

When S is a lattice in Euclidean space and X_o is determined by finite range constraints, it can be quite a difficult combinatorial problem to determine, for a given $\mathfrak{B}$, the connectedness and cycle structure. For systems with one dimensional geometry and constraints of finite range we can give a reasonable geometric expression of the algebraic criteria of the preceding section. By considering aggregates "along the line" the constraints can be considered to be nearest neighbour.

Specifically we consider the case in which $S = Z$, the integers, and Y is a finite set. We assume a function $M : Y \times Y \to R$ with $M(a,b) \geq 0$. Then we define

$$X_o = \{x \in Y^S : M(x^i, x^{i+1}) > 0 \quad \text{for all } i \in S\}.$$

Spaces of this type have received considerable study (see [11]).

The one point conditional probabilities will, in general, not be sufficient to determine the finite set conditional probabilities. We shall show that under a certain condition the j-adjacent point conditional probabilities are sufficient.

3.1. THEOREM. *Let* $\mathfrak{B}$ *denote the collection of all subsets of* S *which consist of* j *adjacent integers. Assume the matrix* M *has the following property:*

$$M^{j+1}(a,c) > 0, \quad M(b,c) > 0 \Rightarrow M^{j}(a,b) > 0 \tag{11}$$

for all $a, b, c \in Y$. *Then*

(i) *If* $x, y \in X_0$ *are connected* mod $\mathcal{A}$, *they are connected* mod $\mathfrak{B}$,

(ii) $\ker \partial_{\mathfrak{B}}$ *is generated by the set of cycles of length* $\leq j + 3$.

Proof. (i) Suppose $x, y \in X_0$ and $x = y \bmod \mathcal{A}$. Let $\ell(x,y) = n-m$ where m and n are respectively the first and last coordinates where x and y differ. If $\ell(x,y) \leq j-1$, then x and y differ by at most j adjacent coordinates so $x = y \bmod \mathfrak{B}$. Otherwise, as $M^{j+1}(x^{n-j}, x^{n+1}) > 0$, $M(y^n, x^{n+1}) > 0$, we have $M^j(x^{n-j}, y^n) > 0$. Thus we can find $w_1, w_2, \ldots, w_{j-1} \in Y$ so that

$$z = (\ldots, x^{m-1}, x^m, \ldots, x^{n-j}, w_1, \ldots, w_{j-1}, y^n, x^{n+1}, \ldots)$$

is an element of X_0. Now x and z differ on at most j adjacent sites, and $\ell(z,y) < \ell(x,y)$. Iteration of the procedure at most $\ell(x,y) - j+1$ times connects x to y mod $\mathfrak{B}$.

(ii) By remark 2.12 it is sufficient to show that $\ker \tilde{\partial}_{\mathfrak{B}}$ is generated by cycles of length $\leq j + 3$. Now since $\ker \tilde{\partial}_{\mathfrak{B}}$ is generated by cycles of the form $c = \sum_{i=1}^{n} (x_i, x_{i+1})$ where $x_i \in X_0$, $1 \leq i \leq n$; $x_1 = x_{n+1}$ and x_i, x_{i+1} differ on at most j adjacent sites, it suffices to prove the result for such cycles. For $x, y \in X_0$, $x = y \bmod \mathcal{A}$ let

$F(x,y)$ = the first site where x and y differ;

$T(x,y)$ = the last site where x and y differ;

$$f(c) = \min_{1 \leq i \leq n} F(x_i, x_{i+1});$$

$$t(c) = \max_{1 \leq i \leq n} T(x_i, x_{i+1});$$

$P(c)$ = least i for which $f(c) = F(x_i, x_{i+1})$;

$Q(c)$ = largest i for which $f(c) = F(x_i, x_{i+1})$.

The aim is to write c as a sum of cycles of length $\leq j + 3$ plus a cycle d with

$$f(c) < f(d) \leq t(d) \leq t(c). \tag{12}$$

After a finite number of iterations of this procedure we have c expressed as the sum of cycles of length $\leq j + 3$.

First we consider the case in which $t(c) - f(c) \leq j - 1$. Then each pair (x_1, x_k), $2 \leq k \leq n$ is equivalent mod α for some $\alpha \in \mathfrak{B}$ so

$$c = \sum_{i=2}^{n-1} \{(x_1, x_i) + (x_i, x_{i+1}) + (x_{i+1}, x_1)\}$$

expresses c as the sum of 3 cycles.

When $t(c) - f(c) \geq j$ we proceed to reduce this difference in two stages. We have $P(c) < Q(c)$ and $x_i^{f(c)} = x_1^{f(c)}$ for $1 \leq i \leq P(c)$ or $Q(c) < i \leq n$, since the least site which changes must eventually return to the original value. The first stage is to write c as the sum of a cycle of length $j + 3$ or less and a cycle d with $Q(d) - P(d) < Q(c) - P(c)$. We repeat this until $Q(d) - P(d) = 1$. The second stage is to express a cycle c with $Q(c) - P(c) = 1$ as the sum of a 3 cycle and a cycle d satisfying (12).

We now consider this second stage, *i.e.* $t(c) - f(c) \geq j$ and $Q(c) - P(c) = 1$. There is no loss of generality in assuming $P(c) = 1$. Then

$$c = \{(x_1, x_2) + (x_2, x_3) + (x_3, x_1)\} + (x_1, x_3) + \sum_{i=3}^{n} (x_i, x_{i+1})$$

gives the required representation, since x_1 and x_3 differ at most on j-adjacent sites. This completes stage two.

For stage one, *i.e.* $Q(c) - P(c) > 1$, we have two cases to consider. For simplicity of notation we assume that $f(c) = 0$.

Case (a). $F(x_2, x_3) \geq j + 1$.

Set $z = (\ldots , x_1^{-1}, x_1^{o}, \ldots , x_1^{j}, x_3^{j+1}, x_3^{j+2}, \ldots)$.

Then $z \in X_o$ and there exist $\alpha, \beta \in \mathfrak{B}$ with $z = x_1 \bmod \alpha$, $z = x_2 \bmod \beta$. Then

$$d = (x_1, z) + (z, x_3) + \sum_{i=3}^{n} (x_i, x_{i+1})$$

satisfies $Q(d) - P(d) < Q(c) - P(c)$ and

$$c = d + (x_1, x_2) + (x_2, x_3) + (x_3, z) + (z, x_1) ,$$

i.e. c is the sum of d and a four cycle.

Case (b). $F(x_2, x_3) \leq j$.

By the method of proof of part (i) we can find $z_1, \ldots , z_k \in X_o$ with $z_1 = x_1$, $z_k = x_3$ and z_i differing from z_{i+1} at most on j-adjacent sites. Also $F(z_i, z_{i+1}) > 0$ for $1 \leq i < k-1$ and $T(z_i, z_{i+1}) \leq t(c)$ for $1 \leq i \leq k$. We can do this with $2 \leq k \leq F(x_2, x_3) + 2$. Then with

$$d = \sum_{i=1}^{k-1} (z_i, z_{i+1}) + \sum_{i=3}^{n} (x_i, x_{i+1})$$

we have $c = d + (x_1, x_2) + (x_2, z_k) + (z_k, z_{k-1}) + \ldots + (z_2, z_1)$.

So c can be expressed as the sum of d and a cycle of length $k + 1 \leq j + 3$. For this d we have $Q(d) - P(d) < Q(c) - P(c)$. This completes the proof.

3.2. *Remark.* Essentially the same proof can be carried out when S is the positive integers instead of all integers.

3.3. *Remark.* Condition (11) of Theorem 3.1 can be replaced by

$$M^{j+1}(a,c) > 0,\ M(a,b) > 0 \Rightarrow M^{j}(b,c) > 0 , \tag{13}$$

with the proof simply reversing the order of certain operations. Any homogeneous finite Markov chain without transient states will satisfy conditions(11) and (13) for sufficiently large j. These conditions and the proof can be adapted to inhomogeneous Markov chains with state spaces varying from site to site.

3.4. *Example.* Let Y be the set of j digit numbers in an arbitrary fixed integer base. Define $M([d_1 d_2 \ldots d_j], [d_2 d_3 \ldots d_j\ e]) = 1$ and $M(a,b) = 0$ otherwise. Since $M^j(a,b) = 1$ for all a, b $\in$ Y, Theorem 3.1 shows that a ratio specification is uniquely determined by its $\mathfrak{B}$ values, with $\mathfrak{B}$ the collection of j-adjacent point subsets of S. Two distinct elements of X_o must differ by at least j sites so knowledge of the ratio specification for sets with j - 1 and fewer elements gives no information about the ratio specification for other sets; the j - 1 point conditional probabilities are trivial.

We now give an illustration of how the algebraic techniques above can be applied in terms of actual conditional probabilities. We use the notation of Theorem 3.1. The topology of $X = Y^S$ is explained in [8], to which we refer the reader for an explanation of the notation $\mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \Lambda^c)$. X_o is a closed subspace of X with the subspace topology.

3.5. THEOREM. *Let the hypothesis of Theorem 3.1 be satisfied. Assume that the real valued continuous function* $P_\alpha(x,y)$ *is given for each* $\alpha \in \mathfrak{B}$ *and* $x = y \bmod \alpha$ *which satisfies* (1) *and* (2) *and* $\sum_x P_\alpha(x,y) = 1$ *for each* $\alpha \in \mathfrak{B}$ *and* $y \in X_o$, *with the sum over those* x *which are equivalent to* y mod α. *Define* $F_\alpha(x,y) = P_\alpha(x,y)/P_\alpha(y,x)$ *and* $F_{\mathfrak{B}}$ *on* $C(\mathfrak{B})$ *by* (8). *Assume* $F_{\mathfrak{B}}$ *is trivial on all elements of* ker $\partial_{\mathfrak{B}}$ *of length* $\leq j + 3$. *If* X_o *is nonempty, then there is a probability measure* μ *on* X_o *such that*

$$\mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \Lambda^c) = P_\Lambda(x,y) \quad \mu\text{. a. e.}$$

for each $\Lambda \in \mathfrak{B}$ *and* $x = y \bmod \Lambda$.

Proof. By Theorems 2.6 and 3.1 $F_{\mathfrak{B}}$ has a unique extension to a ratio specification on $\mathcal{X}$. By Lemma 2.4 we have an equivalence class of algebraic specifications corresponding to $F_{\mathfrak{B}}$. By the requirement that $\sum_x P_\alpha(x,y) = 1$ we have a uniquely defined algebraic specification on $\mathcal{X}$ corresponding to $F_{\mathfrak{B}}$ which coincides for $\alpha \in \mathfrak{B}$ with that originally given. We have continuity for $P_\alpha(x,y)$ since only a finite number of elementary operations are needed to compute it from the given values. By Lemma 5.1 of [6], the $P_\alpha(x,y)$ are consistent. The existence of the required μ follows from Theorem 3.1 of [6].

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